Directed Domination in Oriented Graphs

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Abstract

A directed dominating set in a directed graph D is a set S of vertices of V such that every vertex $u \in V(D) \setminus S$ has an adjacent vertex v in S with v directed to u. The directed domination number of D, denoted by $\gamma(D)$, is the minimum cardinality of a directed dominating set in D. The directed domination number of a graph G, denoted $\Gamma_d(G)$, which is the maximum directed domination number $\gamma(D)$ over all orientations D of G. The directed domination number of a complete graph was first studied by Erdös [Math. Gaz. 47 (1963), 220–222], albeit in disguised form. We extend this notion to directed domination of all graphs. If α denotes the independence number of a graph G, we show that if G is a bipartite graph, we show that $\Gamma_d(G) = \alpha$. We present several lower and upper bounds on the directed domination number.

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1 Introduction

An *asymmetric digraph* or *oriented graph* D is a digraph that can be obtained from a graph G by assigning a direction to (that is, orienting) each edge of G. The resulting digraph D is called an *orientation* of G. Thus if D is an oriented graph, then for every pair u and v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D. A *directed dominating set*, abbreviated DDS, in a directed graph $D = (V, A)$ is a set S of vertices of V such that every vertex in $V \setminus S$ is dominated by some vertex of S; that is, every vertex $u \in V \setminus S$ has an adjacent vertex v in S with v directed to u. Every digraph has a DDS since the entire vertex set of the digraph is such a set. The *directed domination number* of a *directed graph* D, denoted by $\gamma(D)$, is the minimum cardinality of a DDS in D. A DDS of D of cardinality $\gamma(D)$ is called a $\gamma(D)$-set. Directed domination in digraphs is well studied (cf. [2, 3, 6, 7, 8, 12, 15, 19, 22, 23]).

We define the *lower directed domination number* of a *graph* G, denote $\gamma_d(G)$, to be the minimum directed domination number $\gamma(D)$ over all orientations D of G; that is,

$$\gamma_d(G) = \min\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\}.$$

The *upper directed domination number*, or simply the *directed domination number*, of a *graph* G, denoted $\Gamma_d(G)$, is defined as the maximum directed domination number $\gamma(D)$ over all orientations D of G; that is,

$$\Gamma_d(G) = \max\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\}.$$

1.1 Motivation

The directed domination number of a complete graph was first studied by Erdös [11] albeit in disguised form. In 1962, Schütte [11] raised the question of given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ on $n(k)$ vertices in which for any set S of k vertices, there is a vertex u which dominates all vertices in S. Erdös [11] showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer k. The proof of the following bounds on the directed domination number of a complete graph are along identical lines to that presented by Erdös [11]. This result can also be found in [23]. Throughout this paper, log is to the base 2 while ln denotes the logarithm in the natural base e.

Theorem 1 (Erdös [11]) *For every integer* $n \geq 2$, $\log n - 2\log(\log n) \leq \Gamma_d(K_n) \leq \log(n+1)$.

In this paper, we extend this notion of directed domination in a complete graph to directed domination of all graphs.

1.2 Notation

For notation and graph theory terminology we in general follow [18]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. If A and B are subsets of $V(G)$, we let $[A, B]$ denote the set of all edges between A and B in G. We denote the diameter of G by $\mathrm{diam}(G)$.

We denote the *degree* of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree among the vertices of G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. The *maximum average degree* in G, denoted by $\mathrm{mad}(G)$, is defined as the maximum of the average degrees $\mathrm{ad}(H) = 2|E(H)|/|V(H)|$ taken over all subgraphs H of G.

The parameter $\gamma(G)$ denotes the *domination number* of G. The parameters $\alpha(G)$ and $\alpha'(G)$ denote the (vertex) *independence number* and the *matching number*, respectively, of G, while $\chi(G)$ and $\chi'(G)$ denote the *chromatic number* and *edge chromatic number*, respectively, of G. The *covering number* of G, denoted by $\beta(G)$, is the minimum number vertices that covers all the edges of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum cardinality of a clique in G.

A vertex v in a digraph D *out-dominates*, or simply *dominates*, itself as well as all vertices u such that (v, u) is an arc of D. The *out-neighborhood* of v, denoted $N^+(v)$, is the set of all vertices u adjacent from v in D; that is, $N^+(v) = \{u \mid (v, u) \in A(D)\}$. The *out-degree* of v is given by $d^+(v) = |N^+(v)|$, and the maximum out-degree among the vertices of D is denoted by $\Delta^+(D)$. The *in-neighborhood* of v, denoted $N^-(v)$, is the set of all vertices u adjacent to v in D; that is, $N^-(v) = \{u \mid (u, v) \in A(D)\}$. The *in-degree* of v is given by $d^-(v) = |N^-(v)|$. The *closed in-neighborhood* of v is the set $N^-[v] = N^-(v) \cup \{v\}$. The maximum in-degree among the vertices of D is denoted by $\Delta^-(D)$.

A *hypergraph* $H = (V, E)$ is a finite set V of elements, called *vertices*, together with a finite multiset E of subsets of V, called *edges*. A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. For a digraph $D = (V, E)$, we denote by H_D the *closed in-neighborhood hypergraph*, abbreviated CINH, of D; that is, $H_D = (V, C)$ is the hypergraph with vertex set V and with edge set C consisting of the closed in-neighborhoods of vertices of V in D.

2 Observations

We show first that the lower directed domination number of a graph is precisely its domination number.

Observation 1 *For every graph G, $\gamma_d(G) = \gamma(G)$.*

Proof. Let S be a $\gamma(G)$-set and let D be an orientation obtained from G by directing all edges in $[S, V \setminus S]$ from S to $V \setminus S$ and directing all other edges arbitrarily. Then, S is a DDS of D, and so $\gamma_d(G) \le \gamma(D) \le |S| = \gamma(G)$. However if D is an orientation of a graph G such that $\gamma_d(G) = \gamma(D)$, and if S is a $\gamma(D)$-set, then S is also a dominating set of G, and so $\gamma(G) \le |S| = \gamma_d(G)$. Consequently, $\gamma_d(G) = \gamma(G)$. $\Box$

In view of Observation 1, it is not interesting to ask about the lower directed domination number, $\gamma_d(G)$, of a graph G since this is precisely its domination number, $\gamma(G)$, which is very well studied. We therefore focus our attention on the (upper) directed domination number of a graph. As a consequence of Theorem 1, we establish a lower bound on the directed domination number of an arbitrary graph.

Observation 2 *For every graph G on n vertices, $\Gamma_d(G) \ge \log n - 2\log(\log n)$.*

Proof. Let D be an orientation of the edges of a complete graph K_n on the same vertex set as G such that $\Gamma_d(K_n) = \gamma(D)$. Let D_G be the orientation of D induced by arcs of D corresponding to edges of G. Then, $\Gamma_d(G) \ge \gamma(D_G) \ge \gamma(D) = \Gamma_d(K_n)$. The desired lower bound now follows from Theorem 1. $\Box$

Observation 3 *If H is an induced subgraph of a graph G, then $\Gamma_d(G) \ge \Gamma_d(H)$.*

Proof. Let $G = (V, E)$ and let $U = V(H)$. Let D_H be an orientation of H such that $\Gamma_d(H) = \gamma(D_H)$. We now extend the orientation D_H of H to an orientation D of G by directing all edges in $[U, V \setminus U]$ from U to $V \setminus U$ and directing all edges with both ends in $V \setminus U$ arbitrarily. Then, $\Gamma_d(G) \ge \gamma(D) \ge \gamma(D_H) = \Gamma_d(H)$. $\Box$

Observation 4 *If H is a spanning subgraph of a graph G, then $\Gamma_d(G) \le \Gamma_d(H)$.*

Proof. Let D be an arbitrary orientation of G, and let D_H be the orientation of H induced by D. Since adding arcs cannot increase the directed domination number, we have that $\gamma(D) \le \gamma(D_H)$. This is true for every orientation of G. Hence, $\Gamma_d(G) \le \Gamma_d(H)$. $\Box$

Hakimi [17] proved that a graph G has an orientation D such that $\Delta^+(D) \le k$ if and only if $\mathrm{mad}(G) \le 2k$. This implies the following result.

Observation 5 ([17]) *Every graph G has an orientation D such that $\Delta^+(D) \le \lceil \mathrm{mad}(G)/2 \rceil$.*

3 Bounds

In this section, we establish bounds on the directed domination number of a graph. We first present lower bounds on the directed domination number of a graph.

Theorem 2 *Let G be a graph of order n. Then the following holds.*
(a) $\Gamma_d(G) \ge \alpha(G) \ge \gamma(G)$.
(b) $\Gamma_d(G) \ge n/\chi(G)$.
(c) $\Gamma_d(G) \ge \lceil (\mathrm{diam}(G)+1)/2 \rceil$.
(d) $\Gamma_d(G) \ge n/(\lceil \mathrm{mad}(G)/2 \rceil + 1)$.

Proof. Since every maximal independent set in a graph is a dominating set in the graph, we recall that $\gamma(G) \le \alpha(G)$ holds for every graph G. To prove that $\alpha(G) \le \Gamma_d(G)$, let A be a maximum independent set in G and let D be the digraph obtained from G by orienting all arcs from A to $V \setminus A$ and orienting all arcs in $G[V \setminus A]$, if any, arbitrarily. Since every DDS of D contains A, we have $\gamma(D) \ge |A|$. However the set A itself is a DDS of D, and so $\gamma(D) \le |A|$. Consequently, $\Gamma_d(G) \ge \gamma(D) = |A| = \alpha(G)$. This establishes Part (a). Parts (b) and (c) follows readily from Part (a) and the observations that $\alpha(G) \ge n/\chi(G)$ and $\alpha(G) \ge \lceil (\mathrm{diam}(G)+1)/2 \rceil$. By Observations 5, there is an orientation D of G such that $\Delta^+(D) \le \lceil \mathrm{mad}(G)/2 \rceil$. Let S be a $\gamma(D)$-set. Then, $V \setminus S \subseteq \cup_{v \in S} N^+(v)$, and so $n - |S| = |V \setminus S| \le \sum_{v \in S} d^+(v) \le |S| \cdot \Delta^+(D)$, whence $\gamma(D) = |S| \ge n/(\Delta^+(D)+1) \ge n/(\lceil \mathrm{mad}(G)/2 \rceil + 1)$. This establishes Part (d). $\Box$

We remark that since $\mathrm{mad}(G) \le \Delta(G)$ for every graph G, as an immediate consequence of Theorem 2(d) we have that $\Gamma_d(G) \ge n/(\lceil \Delta(G)/2 \rceil + 1)$.

Next we consider upper bounds on the directed domination number of a graph. The following lemma will prove to be useful.

Lemma 3 *Let $G = (V, E)$ be a graph and let $V_1, V_2, \ldots, V_k$ be subsets of V, not necessarily disjoint, such that $\cup_{i=1}^k V_i = V(G)$. For $i = 1, 2, \ldots, k$, let $G_i = G[V_i]$. Then,*

$$\Gamma_d(G) \le \sum_{i=1}^{k} \Gamma_d(G_i).$$

Proof. Consider an arbitrary orientation D of G. For each $i = 1, 2, \ldots, k$, let D_i be the orientation of the edges of G_i induced by D and let S_i be a $\gamma(D_i)$-set. Then, $\Gamma_d(G_i) \ge \gamma(D_i) = |S_i|$ for each i. Since the set $S = \cup_{i=1}^k S_i$ is a DDS of D, we have that $\gamma(D) \le |S| \le \sum_{i=1}^k |S_i| \le \sum_{i=1}^k \Gamma_d(G_i)$. Since this is true for every orientation D of G, the desired upper bound on $\Gamma_d(G)$ follows. $\Box$

As a consequence of Lemma 3, we have the following upper bounds on the directed domination number of a graph.

Theorem 4 *Let G be a graph of order n. Then the following holds.*
(a) $\Gamma_d(G) \le n - \alpha'(G)$.
(b) *If G has a perfect matching, then* $\Gamma_d(G) \le n/2$.
(c) $\Gamma_d(G) \le n$ *with equality if and only if* $G = \overline{K}_n$.
(d) *If G has minimum degree δ and $n \ge 2\delta$, then* $\Gamma_d(G) \le n - \delta$.
(e) $\Gamma_d(G) = n - 1$ *if and only if every component of G is a K_1-component, except for one component which is either a star or a complete graph K_3.*

Proof. (a) Let $M = \{u_1v_1, u_2v_2, \ldots, u_tv_t\}$ be a maximum matching in G, and so $t = \alpha'(G)$. For $i = 1, 2, \ldots, t$, let $V_i = \{u_i, v_i\}$. If $n > 2t$, let $(V_{t+1}, \ldots, V_{n-2t})$ be a partition of the remaining vertices of G into $n - 2t$ subsets each consisting of a single vertex. By Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{n} \Gamma_d(G_i) = t + (n - 2t) = n - t = n - \alpha'(G)$. Part (b) is an immediate consequence of Part (a). Part (c) is an immediate consequence of Part (a) and the observation that $\alpha'(G) = 0$ if and only if $G = \overline{K}_n$.

(d) It is well known (see, for example, Bollobás [4], pp. 87) that if G has n vertices and minimum degree δ with $n \ge 2\delta$, then $\alpha'(G) \ge \delta$. Hence by Part (a) above, $\Gamma_d(G) \le n - \delta$.

(e) Suppose that $\Gamma_d(G) = n - 1$. Then by Part (a) above, $\alpha'(G) = 1$. However every connected graph F with $\alpha'(F) = 1$ is either a star or a complete graph K_3. Hence, either G is the vertex disjoint union of a star and isolated vertices or of a complete graph K_3 and isolated vertices. $\Box$

We establish next that the directed domination number of a bipartite graph is precisely its independence number. For this purpose, recall that König [21] and Egerváry [10] showed that if G is a bipartite graph, then $\alpha'(G) = \beta(G)$. Hence by Gallai's Theorem [13], if G is a bipartite graph of order n, then $\alpha(G) + \alpha'(G) = n$.

Theorem 5 *If G is a bipartite graph, then* $\Gamma_d(G) = \alpha(G)$.

Proof. Since G is a bipartite graph, we have that $n - \alpha'(G) = \alpha(G)$. Thus by Theorem 2(a) and Theorem 4(b), we have that $\alpha(G) \le \Gamma_d(G) \le n - \alpha'(G) = \alpha(G)$. Consequently, we must have equality throughout this inequality chain. In particular, $\Gamma_d(G) = \alpha(G)$. $\Box$

4 Relation to other Parameters

The following result establishes an upper bound on the directed domination of a graph in terms of its independence number and chromatic number.

Theorem 6 *For every graph G, we have $\Gamma_d(G) \le \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.*

Proof. Let G have order n. If $\chi(G) = 1$, then G is the empty graph, $\overline{K}_n$ and so $\Gamma_d(G) = n = \alpha(G)$, while if $\chi(G) = 2$, then G is a bipartite graph, and so by Theorem 5, $\Gamma_d(G) = \alpha(G)$. In both cases, $\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$, and so $\Gamma_d(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$. Hence we may assume that $\chi(G) \ge 3$. If $\chi(G) = 2k$ for some integer $k \ge 2$, then let $V_1, V_2, \ldots, V_{2k}$ denote the color classes of G. For $i = 1, 2, \ldots, k$, let G_i be the subgraph $G[V_{2i-1} \cup V_{2i}]$ of G induced by V_{2i-1} and V_{2i} and note that G_i is a bipartite graph. By Theorem 5, $\Gamma_d(G_i) = \alpha(G_i) \le \alpha(G)$ for all $1, 2, \ldots, k$. Hence by Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{k} \Gamma_d(G_i) \le k\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$, as desired. If $\chi(G) = 2k + 1$ for some integer $k \ge 1$, then let $V_1, V_2, \ldots, V_{2k+1}$ denote the color classes of G. For $i = 1, 2, \ldots, k$, let H_i be the subgraph of G induced by V_{2i-1} and V_{2i} and note that H_i is a bipartite graph. Further let $H_{k+1} = G[V_{2k+1}]$, and so H_{k+1} is an empty graph on $|V_{2k+1}| \le \alpha(G)$ vertices. By Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{k+1} \Gamma_d(H_i) \le (k + 1)\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$. □

As shown in the proof of Theorem 6, the upper bound of Theorem 6 is always attained if $\chi(G) \le 2$. We remark that if $\chi(G) = 3$ or $\chi(G) = 4$, then the upper bound of Theorem 6 is achievable by taking, for example, $G = rK_t$ where $t \in \{3, 4\}$ and r is some positive integer. In this case, $\chi(G) = t$ and $\Gamma_d(G) = 2r = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.

Theorem 7 *If G is a graph of order n, then $\Gamma_d(G) \le n - \lfloor \chi(G)/2 \rfloor$.*

Proof. If $\chi(G) = 1$, then the bound is immediate since $\Gamma_d(G) \le n$ by Theorem 4(c). Hence we may assume that $\chi(G) = k \ge 2$. Let $V_1, V_2, \ldots, V_k$ denote the color classes of G. By the minimality of the coloring, there is an edge between every two color classes. In particular for $i = 1, 2, \ldots, \lfloor k/2 \rfloor$, there is an edge between V_{2i-1} and V_{2i}, and so $\alpha'(G) \ge \lfloor k/2 \rfloor$. Hence by Theorem 4(a), $\Gamma_d(G) \le n - \alpha'(G) \le n - \lfloor k/2 \rfloor$. □

We remark that the bound of Theorem 7 is achievable for graphs with small chromatic number as may be seen by considering the graph $G = \overline{K}_{n-k} \cup K_k$ where $1 \le k \le 4$ and $n > k$. We show next that the directed domination of a graph is at most the average of its order and independence number. For this purpose, we recall the Gallai-Milgram Theorem [14] for oriented graphs which states that in every oriented graph $G = (V, E)$, there is a partition of V into at most $\alpha(G)$ vertex disjoint directed paths.

Theorem 8 *If G is a graph of order n, then $\Gamma_d(G) \le (n + \alpha(G))/2$.*

Proof. Let D be an orientation of G. By the Gallai-Milgram Theorem for oriented graphs, there is a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ of $V(D)$ into t vertex disjoint directed paths where $t \le \alpha(G)$. For $i = 1, 2, \ldots, t$, let $|P_i| = p_i$, and so $\sum_{i=1}^{t} p_i = n$. By Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{t} \Gamma_d(P_i) = \sum_{i=1}^{t} \lceil p_i/2 \rceil \le \sum_{i=1}^{t} (p_i + 1)/2 = (\sum_{i=1}^{t} p_i/2) + t/2 = (n + \alpha(G))/2$. □

That the bound of Theorem 8 is best possible, may be seen by considering, for example, the graph $G = rK_3 \cup sK_1$ of order $n = 3r + s$ with $\alpha(G) = r + s$ and $\Gamma_d(G) = 2r + s = (n + \alpha(G))/2$.

The following result establishes an upper bound on the directed domination of a graph in terms of the chromatic number of its complement.

Theorem 9 *If G is a graph of order n, then* $\Gamma_d(G) \le \chi(\overline{G}) \cdot \log\left(\left\lceil \frac{n}{\chi(\overline{G})} \right\rceil + 1\right)$.

Proof. Let $t = \chi(\overline{G})$ and consider a $\chi(\overline{G})$-coloring of the complement $\overline{G}$ of G into t color classes $Q_1, Q_2, \ldots, Q_t$, where $|Q_i| = q_i$ for $i = 1, 2, \ldots, t$. For each $i = 1, 2, \ldots, t$, the subgraph $G[Q_i]$ of G induced by Q_i is a clique. We now consider an arbitrary orientation D of G, and we let $D_i = D[Q_i]$ denote the orientation of the edges of the clique $G[Q_i]$ induced by D. Then,

$$\gamma(D) \le \sum_{i=1}^{t} \gamma(D_i) \le \sum_{i=1}^{t} \Gamma_d(Q_i) = \sum_{i=1}^{t} \Gamma_d(K_{q_i}).$$

This is true for every orientation D of G, and so, by Theorem 1, we have that $\Gamma_d(G) \le \sum_{i=1}^{t} \log(q_i + 1)$, where $\sum_{i=1}^{t} q_i = n$. By convexity the right hand side attains its maximum when all summands are as equal as possible; that is, some of the summands are $\lfloor n/t \rfloor$ and some are $\lceil n/t \rceil$. Hence, $\Gamma_d(G) \le t \log(\lceil n/t \rceil + 1)$. $\Box$

As a consequence of Theorem 9, we have the following result on the directed domination number of a dense graph with large minimum degree.

Theorem 10 *If G is a graph on n vertices with minimum degree $\delta(G) \ge (k-1)n/k$ where k divides n, then $\Gamma_d(G) \le n \log(k+1)/k$.*

Proof. Since $k \,|\, n$, we note that $n = kt$ and $\delta(G) \ge (k-1)t$ for some integer t. By the well-known Hajnal-Szemerédi Theorem [16], the graph G contains t vertex disjoint copies of K_k. Further, $\chi(\overline{G}) \le t$. Thus applying Theorem 9, we have that $\Gamma_d(G) \le t \log(k+1) = n \log(k+1)/k$. $\Box$

5 Special Families of Graphs

In this section, we consider the (upper) directed domination number of special families of graph. As remarked earlier, the directed domination number of a complete graph K_n is determined by Erdös [11] in Theorem 1, while the directed domination number of a bipartite graph is precisely its independence number (see Theorem 5).

5.1 Regular Graphs

For each given $\delta \ge 1$, applying Theorem 2(a) to the graph $G = K_{\delta, n-\delta}$ yields $\Gamma_d(G) \ge n - \delta$. Hence without regularity, we observe that for each fixed $\delta \ge 1$, there exists a graph G of

order n and minimum degree δ satisfying $\Gamma_d(G) \geq n - \delta$. With regularity, the directed domination number of a graph may be much smaller. For a given r, let $n = k(r+1)$ for some integer k and let G consist of the disjoint union of k copies of K_{r+1}. Let $G_1, G_2, \ldots, G_k$ denote the components of G. Each component of G is r-regular, and by Theorem 1, $\Gamma_d(G) = \sum_{i=1}^{k} \Gamma_d(G_i) = \sum_{i=1}^{k} \Gamma_d(K_{r+1}) \leq k \log(r+2) = n \log(r+2)/(r+1)$. Hence there exist r-regular graphs of order n with $\Gamma_d(G) \leq n \log(r+2)/(r+1)$. In view of these observations it is of interest to investigate the directed domination number of regular graphs.

In 1964, Vizing proved his important edge-coloring result which states that every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. As a consequence of Vizing's Theorem, we have the following upper bound on the directed domination number of a regular graph.

Theorem 11 *For $r \geq 2$, if G is an r-regular graph of order n, then*

$$\Gamma_d(G) \leq n(r+2)/2(r+1).$$

Proof. By Vizing's Theorem, $\chi'(G) \leq r+1$. Consider an edge coloring of G using $\chi'(G)$-colors. The edges in each color class form a matching in G, and so the matching number of G is at least the size of a largest color class in G. Hence if G has size m, we have $\alpha'(G) \geq m/\chi'(G) \geq m/(r+1) = nr/2(r+1)$. Hence by Theorem 4(a), $\Gamma_d(G) \leq n - \alpha'(G) \leq n - nr/2(r+1) = n(r+2)/2(r+1)$. $\Box$

As a special case of Theorem 11, we have that $\Gamma_d(G) \leq 2n/3$ if G is a 2-regular graph. We next characterize when equality is achieved in this bound.

Proposition 1 *Let G be a 2-regular graph on $n \geq 3$ vertices. Then the following holds.*
(a) *If G is connected, then $\Gamma_d(G) = \lceil n/2 \rceil$.*
(b) *$\Gamma_d(G) \leq 2n/3$ with equality if and only if G consists of disjoint copies of K_3.*

Proof. (a) Suppose that G is a cycle C_n. If n is even, G has a perfect matching, and so, by Theorem 4(c), $\Gamma_d(G) \leq n/2$. If n is odd, then $\alpha'(G) = (n-1)/2$. By Theorem 4(b), $\Gamma_d(G) \leq n - \alpha'(G) = n - (n-1)/2 = (n+1)/2$. In both cases, $\Gamma_d(G) \leq \lceil n/2 \rceil$. To show that $\Gamma_d(G) \geq \lceil n/2 \rceil$, we note that if D is a directed cycle C_n, then every vertex out-dominates itself and exactly one other vertex, and so $\Gamma_d(G) \geq \gamma(D) = \lceil n/2 \rceil$. This proves part (a).

(b) To prove part (b), let $G_1, G_2, \ldots, G_k$ be the components of G, where $k \geq 1$. For $i = 1, 2, \ldots, k$, let G_i have order n_i. Since each component of a cycle, $n \geq 3k$. Applying the result of part (a) to each component of G, we have

$$\Gamma_d(G) = \sum_{i=1}^{k} \Gamma_d(G_i) \leq \sum_{i=1}^{k} \left(\frac{n_i + 1}{2} \right) = \frac{n+k}{2} \leq \frac{2n}{3},$$

with equality if and only if $n = 3k$, i.e., if and only if $G_i = C_3$ for each $i = 1, 2, \ldots, k$. $\Box$

We remark that the upper bound of Theorem 11 can be improved using tight lower bounds on the size of a maximum matching in a regular graph established in [20]. Applying Theorem 4(a) to these matching results in [20], we have the following result. We remark that the $(n+1)/2$ bound in the statement of Theorem 12 is only included as it is necessary when n is very small or $r = 2$.

Theorem 12 *For $r \geq 2$, if G is a connected r-regular graph of order n, then*

$$\Gamma_d(G) \leq \begin{cases} \max\left\{ \left(\dfrac{r^2+2r}{r^2+r+2}\right) \times \dfrac{n}{2}, \dfrac{n+1}{2} \right\} & \text{if } r \text{ is even} \\[2ex] \dfrac{(r^3+r^2-6r+2)\,n+2r-2}{2(r^3-3r)} & \text{if } r \text{ is odd} \end{cases}$$

We close this section with the following observation. Graphs G satisfying $\chi'(G) = \Delta(G)$ are called *class 1* and those with $\chi'(G) = \Delta(G) + 1$ are *class 2*.

Observation 6 *Let G be an r-regular graph of order n. Then the following holds.*
(a) *If G is of class 1, then $\Gamma_d(G) \leq n/2$.*
(b) *If $r \geq n/2$, then $\Gamma_d(G) \leq \lceil n/2 \rceil$.*

Proof. (a) Consider a r-edge coloring of G. The edges in each color class form a perfect matching in G, and so, by Theorem 4(c), $\Gamma_d(G) \leq n/2$.

(b) If $n = 2$, then the result is immediate. Hence we may assume that $n \geq 3$. By Dirac's theorem, G is hamiltonian, and so $\alpha'(G) \geq \lfloor n/2 \rfloor$. By Theorem 4(b), $\Gamma_d(G) \leq n - \alpha'(G) \leq n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$. $\square$

5.2 Outerplanar Graphs

Let $\mathcal{OP}_n$ denote the family of all maximal outerplanar graphs of order n. We define $\mathrm{Mop}(n) = \max\{\Gamma_d(G)\}$ where the maximum is taken over all graphs $G \in \mathcal{OP}_n$.

Theorem 13 $\mathrm{Mop}(n) = \lceil n/2 \rceil$.

Proof. Let $G \in \mathcal{OP}_n$. Since every maximal outerplanar graph is hamiltonian, we observe by Observation 4 and Proposition 1(a), that $\Gamma_d(G) \leq \Gamma_d(C_n) = \lceil n/2 \rceil$. Since this is true for an arbitrary graph G in $\mathcal{OP}_n$, we have $\mathrm{Mop}(n) \leq \lceil n/2 \rceil$. Hence it suffices for us to prove that $\mathrm{Mop}(n) \geq \lceil n/2 \rceil$. If $n = 3$, then by Observation 3, $\Gamma_d(G) \geq \Gamma_d(C_n) = \lceil n/2 \rceil$, as desired. Hence we may assume that $n \geq 4$, for otherwise the desired result follows.

For $n \geq 4$ even, we take a directed cycle $\overrightarrow{C_n}$ on $n \geq 4$ vertices and a selected vertex v on the cycle, and we add arcs from every vertex u, where u is neither the in-neighbor nor the out-neighbor of v on $\overrightarrow{C_n}$, to the vertex v. The resulting orientation D of the underlying maximal outerplanar graph has $\gamma_d(D) = n/2$. Hence for $n \geq 4$ even, we have $\mathrm{Mop}(n) = n/2$.

It remains for us to show that for $n \geq 5$ odd, $\mathrm{Mop}(n) = (n+1)/2$. For $n \geq 5$ odd, we take a directed cycle $\overrightarrow{C_n}$: $v_1 v_2 \ldots v_n v_1$ on n vertices. We now add the arcs from v_i to v_1 for all odd i, where $3 \leq i \leq n-2$, and we add the arcs from v_1 to v_i for all even i, where $4 \leq i \leq n-1$. Let G denote the resulting underlying maximal outerplanar graph and let D denote the resulting orientation of D. We now consider an arbitrary DDS S in D.

Suppose first that $v_1 \in S$. In order to dominate the $(n-1)/2$ vertices v_{2i+1}, where $1 \leq i \leq (n-1)/2$, in D we must have that $|S \cap \{v_{2i}, v_{2i+1}\}| \geq 1$ for all $i = 1, 2, \ldots, (n-1)/2$. Hence in this case when $v_1 \in S$, we have $|S| \geq (n+1)/2$.

Suppose next that $v_1 \notin S$. Then, $v_2 \in S$. In order to dominate the $(n-3)/2$ vertices v_{2i}, where $2 \leq i \leq (n-1)/2$, in D we must have that $|S \cap \{v_{2i}, v_{2i-1}\}| \geq 1$ for all $i = 2, \ldots, (n-1)/2$. In order to dominate v_1, there is a vertex $v_j \in S$ for some odd j, where $3 \leq j \leq n$. Let j be the largest such odd subscript for which $v_j \in S$. If $j = n$, then $v_n \in S$ and $|S| \geq (n+1)/2$, as desired. Hence we may assume that $j < n$. In order to dominate the vertex v_i for i odd with $j < i \leq n$, we must have $v_{i-1} \in S$. In particular, we have that $v_{j+1} \in S$ to dominate v_{j+2}, implying that $|S \cap \{v_j, v_{j+1}\}| = 2$ while for i odd where $i \neq j$ and $3 \leq i \leq n-2$, we have $|S \cap \{v_i, v_{i+1}\}| \geq 1$, implying that $|S| \geq (n+1)/2$.

In both cases, $|S| \geq (n+1)/2$. Since S is an arbitrary DDS in D, we have $\gamma(D) \geq (n+1)/2$. Hence, $\Gamma_d(G) \geq (n+1)/2$, implying that $\mathrm{Mop}(n) = (n+1)/2$. $\square$

5.3 Perfect Graphs

Recall that a *perfect graph* is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. Characterization of perfect graphs was a longstanding open problem. The first breakthrough was due to Lovsz in 1972 who proved the Perfect Graph Theorem.

Perfect Graph Theorem *A graph is perfect if and only if its complement is perfect.*

Let $\alpha \geq 1$ be an integer and let $\mathcal{G}_\alpha$ be the class of all graphs G with $\alpha \geq \alpha(G)$. We are now in a position to present an upper bound on the directed domination number of a perfect graph in terms of its independence number.

Theorem 14 *If $G \in \mathcal{G}_\alpha$ is a perfect graph of order $n \geq \alpha$, then*

$$\Gamma_d(G) \leq \alpha \log\left(\lceil n/\alpha \rceil + 1\right).$$

Proof. By the Perfect Graph Theorem, the complement $\overline{G}$ of G is perfect. Hence, $\chi(\overline{G}) = \omega(\overline{G}) = \alpha(G)$. The desired result now follows from Theorem 9. $\Box$

6 Interplay between Transversals and Directed Domination

In this section, we present upper bounds on the directed domination number of a graph by demonstrating an interplay between the directed domination number of a graph and the transversal number of a hypergraph. We shall need the following upper bounds on the transversal number of a uniform hypergraph established by Alon [1] and Chvátal and McDiarmid [9]. Applying probabilistic arguments, Alon [1] showed the following result.

Theorem 15 (Alon [1]) *For $k \geq 2$, if H is a k-uniform hypergraph with n vertices and m edges, then $\tau(H) \leq (m+n)(\ln k)/k$.*

Theorem 16 (Chvátal, McDiarmid [9]) *For $k \geq 2$, if H is a k-uniform hypergraphs with n vertices and m edges, then $\tau(H) \leq (n + \lfloor \frac{k}{2} \rfloor m)/\lfloor \frac{3k}{2} \rfloor$. bound is sharp.*

We proceed further with two lemmas. For this purpose, we shall need the Szekeres-Wilf Theorem.

Theorem 17 (Szekeres-Wilf [24]) *If G is a k-degenerate graph, then $\chi(G) \leq k+1$.*

Lemma 18 *If G is a graph and D is an orientation of G such that $\Delta^-(D) \leq k$ for some fixed integer $k \geq 0$, then $\chi(G) \leq 2k+1$.*

Proof. It suffices to show that G is $2k$-degenerate, since then the desired result follows from the Szekeres-Wilf Theorem. Assume, to the contrary, that G is not $2k$-degenerate. Then there is a subset S of $V(G)$ such that the subgraph $G_S = G[S]$ induced by S has minimum degree at least $2k+1$ and hence contains at least $(2k+1)|S|/2$ edges. Let $D_S = D[S]$ be the orientation of D induced by S. Since $\Delta^-(D) \leq k$, we have that $\Delta^-(D_S) \leq k$ and

$$k|S| \geq \sum_{v \in V(D_S)} d^-(v) = |E(G_S)| \geq (2k+1)|S|/2 > k|S|,$$

a contradiction. $\Box$

Lemma 19 *Let D be an orientation of a graph G. If G contains n_k vertices with in-degree at most k in D for some fixed integer $k \geq 0$, then $n_k \leq (2k+1)\alpha(G)$.*

Proof. Let V_k denote the set of all vertices of G with in-degree at most k in D, and so $n_k = |V_k|$. Let $G_k = G[V_k]$ and let $D_k = D[V_k]$. Then, D_k is an orientation of G_k

such that $\Delta^-(D_k) \le k$, and so by Lemma 18, $\chi(G_k) \le 2k+1$. Since every color class of G_k is an independent set, and therefore has cardinality at most $\alpha(G)$, we have that $n_k = |V_k| \le \chi(G_k)\alpha(G) \le (2k+1)\alpha(G)$. $\square$

Let $f(n,k)$, $g(n,k)$, and $h(n,k)$ be the functions of n and k defined as follows.

$$\begin{aligned} f(n,k) &= 2n\ln(k+2)/(k+2) + (2k+1)\alpha(G) \\ g(n,k) &= n(k+2)/3k + 2(2k+1)\alpha(G)/3 \\ h(n,k) &= n(k+1)/(3k-1) + 2k(2k+1)\alpha(G)/(3k-1) \end{aligned}$$

Theorem 20 *If G is a graph on n vertices, then*

$$\Gamma_d(G) \le \begin{cases} \min\limits_{k\ge 0}\{f(n,k), g(n,k)\} & \textit{if } k \textit{ is even} \\ \\ \min\limits_{k\ge 1}\{f(n,k), h(n,k)\} & \textit{if } k \textit{ is odd} \end{cases}$$

Proof. Let D be an arbitrary orientation of the graph G and let $k \ge 0$ be an arbitrary integer. Let V_k denote the set of all vertices of G with in-degree at most k in D and let $n_k = |V_k|$. Let $V_{>k} = V(G) \setminus V_k$, and so all vertices in $V_{>k}$ have in-degree at least $k+1$ in D. Let $H_{>k}$ be the hypergraph obtained from the CINH H_D of D by deleting the n_k edges corresponding to closed in-neighborhoods of vertices in V_k. Each edge in $H_{>k}$ has size at least $k+2$.

We now define the hypergraph H as follows. For each edge e_v in $H_{>k}$ corresponding to the closed in-neighborhood of a vertex v in $V_{>k}$, let e'_v consist of v and exactly $k+1$ vertices from $N^-(v)$. Thus, $e'_v \subseteq e_v$ and e'_v has size $k+2$. Let H be the hypergraph obtained from $H_{>k}$ by shrinking all edges e_v of $H_{>k}$ to the edges e'_v. Then, H is a $(k+2)$-uniform hypergraph with n vertices and $n - n_k$ edges.

Every transversal T in H contains a vertex from the closed in-neighborhood of each vertex from the set $V_{>k}$ in D, and therefore $T \cup V_k$ is a DDS in D. In particular, taking T to be a minimum transversal in H, we have that $\gamma(D) \le \tau(H) + n_k$. By Lemma 19, $n_k \le (2k+1)\alpha(G)$. Applying Theorem 15 to the hypergraph H, we have that

$$\tau(H) \le (n + n - n_k)\ln(k+2)/(k+2) \le 2n\ln(k+2)/(k+2),$$

and so $\gamma(D) \le \tau(H) + n_k \le 2n\ln(k+2)/(k+2) + \alpha(G)(2k+1) = f(n,k)$. Applying Theorem 16 to the hypergraph H for k even, we have that

$$\tau(H) \le (2n + k(n - n_k))/3k = n(k+2)/3k - n_k/3,$$

and so $\gamma(D) \le \tau(H) + n_k \le n(k+2)/3k + 2n_k/3 \le n(k+2)/3k + 2(2k+1)\alpha(G)/3 = g(n,k)$. Thus for k even, we have that $\Gamma_d(G) \le \min\{f(n,k), g(n,k)\}$. Applying Theorem 16 to the hypergraph H for k odd, we have that

$$\tau(H) \le (2n + (k-1)(n - n_k))/(3k-1) = n(k+1)/(3k-1) - (k-1)n_k/(3k-1),$$

and so $\gamma(D) \le \tau(H) + n_k \le n(k+1)/(3k-1) + 2kn_k/(3k-1) \le n(k+1)/(3k-1) + 2k(2k+1)\alpha(G)/(3k-1) = h(n,k)$. Thus for k odd, we have that $\Gamma_d(G) \le \min\{f(n,k), h(n,k)\}$. $\Box$

Let $f_n(\alpha)$, $g_n(\alpha)$, and $h_n(\alpha)$ be the functions of n and α defined as follows.

$$\begin{aligned}
f_n(\alpha) &\doteq \sqrt{2n\alpha}\left(\ln(\sqrt{2n/\alpha}\,) + 2\right) - 2\alpha \\
g_n(\alpha) &\doteq \frac{1}{3}\left(n + 2\alpha + 4\sqrt{2n\alpha}\right) \\
h_n(\alpha) &\doteq \frac{1}{3}\left(n + \frac{14}{3}\alpha + \frac{\sqrt{2\alpha}\,(27n + 20\alpha)}{3\sqrt{5\alpha + 6n}}\right)
\end{aligned}$$

As a consequence of Theorem 20, we have the following upper bound on the directed domination of a graph.

Theorem 21 *If G is a graph on n vertices with independence number α, then*

$$\Gamma_d(G) \le \min\{f_n(\alpha), g_n(\alpha), h_n(\alpha)\}.$$

Proof. By Theorem 20, we need to optimize the functions $f(n,k), g(n,k)$ and $h(n,k)$ over k to obtain an upper bound on $\Gamma_d(G)$. To simplify the notation, let $\alpha = \alpha(G)$. Optimizing the function $g(n,k)$ over k (treating n as fixed), we get $g(n,k) \le g_n(\alpha)$, while optimizing the function $h(n,k)$ over k (treating n as fixed), we get $h(n,k) \le h_n(\alpha)$. Optimization of the function $f(n,k)$ is complicated. Hence to simplify the computations, we choose a value k^* for k and show that $f(n,k^*) \le f_n(\alpha)$. Suppose $\alpha \ge n/2$. Then, $\alpha = cn$ with $1 \ge c \ge 1/2$. Substituting this into $f_n(\alpha)$ we get $f_n(\alpha) = n\sqrt{2c}(\ln(2/c) + 2) - 2cn = n\left(\sqrt{2c}(\ln(2/c) + 2) - 2c\right) \ge n$, and so the inequality $\Gamma_d(G) \le f_n(\alpha)$ holds trivially. Hence we may assume that $\alpha \le n/2$. We now take $k = \sqrt{2n/\alpha} - 2 \ge 0$. Substituting into $f(n,k) = 2n\ln(k+2)/(k+2) + (2k+1)\alpha$, we get

$$\begin{aligned}
f(n,k) &= 2n\ln(\sqrt{2n/\alpha}\,)/\sqrt{2n/\alpha} + (2\sqrt{2n/\alpha} - 3)\alpha \\
&= \sqrt{2n\alpha}\ln(\sqrt{2n/\alpha}\,) + 2\alpha\sqrt{2n/\alpha} - 3\alpha \\
&= \sqrt{2n\alpha}\left(\ln(\sqrt{2n/\alpha}\,) + 2\right) - 3\alpha \\
&< f_n(\alpha),
\end{aligned}$$

as desired. $\Box$

If every edge of a hypergraph H has size at least r, we define an *r-transversal* of H to be a transversal T such that $|T \cap e| \ge r$ for every edge e in H. The *r-transversal number* $\tau_r(H)$ of H is the minimum size of an r-transversal in H. In particular, we note that $\tau_1(H) = \tau(H)$. For integers $k \ge r$ where $k \ge 2$ and $r \ge 1$, we first establish general upper bounds on the r-transversal number of a k-uniform hypergraph. Our next result generalizes that of Theorem 15 due to Alon [1], as well as generalizes results due to Caro [5].

Theorem 22 *For integers $k \geq r$ where $k \geq 2$ and $r \geq 1$, let H be a k-uniform hypergraph with n vertices and m edges. Then, $\tau_r(H) \leq n \ln k/k + rm(2\ln k)^r/k$.*

Proof. Pick every vertex of $V(H)$ randomly with probability p to be determined later but such that $(1-p) > 1/2$. Let X be the set of randomly picked vertices and let E_X be the set of edges of $E(H)$ whose intersection with X is at most $r-1$. For every fixed edge $e \in E(H)$, the probability that e is in E_X is exactly

$$\Pr(e \in E_X) = \sum_{i=0}^{r-1} \binom{k}{i} p^i (1-p)^{k-i} = (1-p)^k \sum_{i=0}^{r-1} \binom{k}{i} \left(\frac{p}{1-p}\right)^i. \tag{1}$$

We now choose $p = \ln k/k$. With this choice of p, we have that $(1-p) > 1/2$. Hence, $1/(1-p)^i < 2^i$ for all $i \geq 1$. Since $1 - x \leq e^{-x}$ for all $x \in R$, we note that $(1-p)^k \leq e^{-pk} = e^{-\ln k} = 1/k$. Substituting $p = \ln k/k$ into Equation (1) we therefore get

$$\Pr(e \in E_X) \leq \frac{1}{k} \sum_{i=0}^{r-1} \frac{k^i}{i!} \cdot \frac{p^i}{(1-p)^i} \leq \frac{1}{k} \sum_{i=0}^{r-1} \frac{(2kp)^i}{i!} \leq \frac{1}{k} \sum_{i=0}^{r-1} (2\ln k)^i \leq \frac{1}{k}(2\ln k)^r,$$

since $1 + q + q^2 + \cdots + q^{r-1} = (q^r - 1)/(q-1) \leq q^r$ for $q > 1$ and $r \geq 1$. For each edge $e \in E_X$, we add $r - |e \cap X|$ (which is at most r) vertices from $e \setminus X$ to a set Y. Then, $T = X \cup Y$ is a r-transversal in H and $|Y| \leq r|E_X|$. By the linearity of expectation, $E(T) = E(X) + E(Y) \leq E(X) + rE(E_X) = n \ln k/k + rm(2\ln k)^r/k$. $\Box$

Using r-transversals in hypergraphs, we obtain the following bound on the directed r-domination number of a graph.

Theorem 23 *For $r \geq 1$ an integer, if G is a graph on n vertices, then*

$$\Gamma_d(G, r) \leq \min_{k \geq r} \left\{(2k-1)\alpha(G) + n\ln(k+1)/(k+1) + rn(2\ln(k+1))^r/(k+1)\right\}.$$

Proof. Let D be an arbitrary orientation of the graph G and let $k \geq r$ be an arbitrary integer. Let $V_{<k}$ denote the set of all vertices of G with in-degree at most $k-1$ in D and let $n_{<k} = |V_{<k}|$. Let $G_{<k}$ be the subgraph of G induced by the set $V_{<k}$ and let $D_{<k}$ be the orientation of $G_{<k}$ induced by D. Then, $\Delta^-(D_{<k}) \leq k-1$, and so, by Lemma 18, $\chi(G_{<k}) \leq 2k-1$, implying that $n_{<k} \leq (2k-1)\alpha(G)$.

Let $V_k = V(G) \setminus V_{<k}$, and so all vertices in V_k have in-degree at least k in D. Let H_k be the hypergraph obtained from the CINH H_D of D by deleting the $n_{<k}$ edges corresponding to closed in-neighborhoods of vertices in $V_{<k}$. Each edge in H_k has size at least $k+1$. We now define the hypergraph H as follows. For each edge e_v in H_k corresponding to the closed in-neighborhood of a vertex v in V_k, let e'_v consist of v and exactly k vertices from

$N^-(v)$. Thus, $e'_v \subseteq e_v$ and e'_v has size $k+1$. Let H be the hypergraph obtained from H_k by shrinking all edges e_v of H_k to the edges e'_v. Then, H is a $(k+1)$-uniform hypergraph with n vertices and $n - n_{<k}$ edges.

Every r-transversal T in H contains at least r vertices from the closed in-neighborhood of each vertex from the set V_k in D, and therefore $T \cup V_{<k}$ is a DrDS in D. In particular, taking T to be a minimum r-transversal in H, we have that $\gamma_r(D) \le \tau_r(H) + n_{<k}$. By Lemma 19, $n_{<k} \le (2k-1)\alpha(G)$. Noting that $k+1 \ge r+1 \ge 2$, we can apply Theorem 22 to the hypergraph H yielding $\tau_r(H) \le n\ln(k+1)/(k+1) + r(n-n_{<k})(2\ln(k+1))^r/(k+1)$, and so $\gamma_r(D) \le \tau_r(H) + n_{<k} \le (2k-1)\alpha(G) + n\ln(k+1)/(k+1) + rn(2\ln(k+1))^r/(k+1)$. Since this is true for every integer $k \ge r$, the desired upper bound on $\Gamma_d(G,r)$ follows. $\Box$

7 Open Questions

We close with a list of open questions and conjectures that we have yet to settle. Let $\mathcal{R}_n$ denote the family of all r-regular graphs of order n. We define $m(n,r) = \min\{\Gamma_d(G)\}$ and $M(n,r) = \max\{\Gamma_d(G)\}$, where the minimum and maximum are taken over all graphs $G \in \mathcal{R}_n$. Then, $m(n,1) = M(n,1) = n/2$. By Proposition 1, $m(n,2) = n/2$ while $M(n,2) = 2n/3$. We remark that by Theorem 11, for $r \ge 2$, we know that

$$\frac{n}{2} \le M(n,r) \le \left(\frac{r+2}{r+1}\right) \cdot \frac{n}{2} \tag{2}$$

(and this upper bound on $M(n,r)$ can be improved slightly by Theorem 12).

Conjecture 1. For $r \ge 3$, $M(n,r) = n/2$.

By Theorem 2(a), we know that if $G \in \mathcal{R}_n$, then $\Gamma_d(G) \ge \alpha(G) \ge n/(r+1)$, and so $n/(r+1) \le m(n,r)$. Moreover taking $n/(r+1)$ copies of K_{r+1}, we have by Theorem 1 that $m(n,r) \le n\log(r+2)/(r+1)$. We pose the following question.

Question 1. For $r \ge 3$, does there exists a constant c such that $m(n,r) \le cn/(r+1)$?

Let $\mathcal{OP}_n$ denote the family of all maximal outerplanar graphs of order n and define $\mathrm{mop}(n) = \min\{\Gamma_d(G)\}$, where the minimum is taken over all graphs $G \in \mathcal{OP}_n$. Since outerplanar graphs are 3-colorable, we note by Theorem 2(b) that for every graph $G \in \mathcal{OP}_n$, $\Gamma_d(G) \ge n/3$, implying that $\mathrm{mop}(n) \ge n/3$. By Theorem 13, we know that $\mathrm{mop}(n) \le \lceil n/2 \rceil$. Thus, $n/3 \le \mathrm{mop}(n) \le \lceil n/2 \rceil$.

Problem 1. Find good lower and upper bounds on $\mathrm{mop}(n)$.

Let $\mathcal{P}_n$ denote the family of all maximum planar graphs of order n. We define $\mathrm{mp}(n) = \min\{\Gamma_d(G)\}$ and $\mathrm{Mp}(n) = \max\{\Gamma_d(G)\}$, where the minimum and maximum are taken over all graphs $G \in \mathcal{P}_n$.

Problem 2. Find good lower and upper bounds on $\mathrm{mp}(n)$ and $\mathrm{Mp}(n)$.

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